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Mild Almost Periodic Solutions of Abstract Differential Equations

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Necessary and sufficient conditions are established for the existence of mild almost periodic solutions of Cauchy problems $u' = Au + f$ and $u'' = Au + f$, where A is the infinitesimal generator of a strongly continuous semigroup (respectively, a cosine function) on a Hilbert space and f an almost periodic function.

INTRODUCTION

Let H be a Hilbert space and A the infinitesimal generator of a strongly continuous semigroup T on H. The problem of the existence of periodic solutions of the equation

$$u' = Au + f,$$

where f is a periodic H-valued function, has been studied by many authors. In particular, in [7] several results of this kind are presented.

Mild solutions on the whole real line for equations with generators of semigroups were first considered in [13], which extends results in [6].

In the first section of this paper, Haraux's [7] results are extended to almost periodic functions.

Using these results, a general characterization is obtained in the case where the operator A is perturbed by a relatively bounded operator which satisfies certain spectral properties.

In the second section, the second order Cauchy problem is studied. It is known [5] that this study cannot be covered, in general, by going back to first order problems; however, characterizations analogous to those of the first section are obtained.

Finally, applications to the Schrödinger equation and to the wave equation are considered.

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1. The Case of the First Order Problem

We will denote by H a Hilbert space with norm $\| \ \|$. Recall [8] that Bohr's transform of an almost periodic function $f\colon \mathbb{R} \to H$ is defined by the expression

$$a(\lambda, f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-i\lambda t} f(t)\, dt.$$

We use the notation

$$\begin{aligned} \mathrm{BC}(\mathbb{R}, H) &= \{f\colon \mathbb{R} \to H \mid f \text{ is continuous} \\ &\qquad \text{and } \|f\| = \sup_{t \in \mathbb{R}} \|f(t)\| < \infty\} \\ \mathrm{AP}(\mathbb{R}, H) &= \{f \in \mathrm{BC}(\mathbb{R}, H) \mid f \text{ is almost periodic (a.p.)}\} \\ \sigma(f) &= \{\lambda \in \mathbb{R} \mid a(\lambda, f) \neq 0\} \end{aligned}$$

It is well known that the set $\sigma(f)$ is, at most, denumerable (see [1]).

We will denote by Λ a denumerable set $\{\lambda_n \in \mathbb{R}, n \geqq 0\}$ such that:

(i) $\lim_{n \to \infty} |\lambda_n| = +\infty$.

(ii) There exists a constant $\Theta \in (0, 1)$ such that $|\lambda_n| \leqq \Theta\, |\lambda_{n+1}|$.

We set

$$\mathrm{AP}_\Lambda(\mathbb{R}, H) = \{f \in \mathrm{AP}(\mathbb{R}, H) \mid \sigma(f) = \Lambda\}.$$

Let f in $L^1(\mathbb{R}, H)$ be fixed, and let A be the infinitesimal generator of a C_0-semigroup on H; a function u in $C(\mathbb{R}, H)$ is called a mild solution (on $\mathbb{R}$) of

$$u'(t) = Au(t) + f(t) \tag{1.1}$$

if the following formula is valid:

$$u(t) = T_{t-s} u(s) + \int_s^t T_{t-v} f(v)\, dv, \qquad s \leqq t, s \in \mathbb{R} \tag{1.2}$$

(see [4, 11, 7]).

If $u \in C^1(\mathbb{R}, H)$ and satisfies (1.1), then it is called a strict solution.

Conditions under which a mild solution is a strict solution have been established by several authors; see [4, 11].

The following result, valid also in Banach spaces, will be fundamental in our considerations. Its proof appears in [12] and generalizes [7].

THEOREM 1.1. *Let A be the infinitesimal generator of a C_0-semigroup on a Hilbert space H. Let $f \in \mathrm{AP}(\mathbb{R}, H)$ and let u be a mild* a.p. *solution of* (1.1). *Then $a(\lambda, u) \in D(A)$ and*

$$(i\lambda - A)\, a(\lambda, u) = a(\lambda, f), \qquad \lambda \in \mathbb{R}. \tag{1.3}$$

THEOREM 1.2. *Let A be the infinitesimal generator of a C_0-semigroup on a Hilbert space H. Suppose that $i\Lambda \subseteq \rho(A)$ and that there is a sequence $\{t_n\}_{n \geqq 0}$ in $l^2(\mathbb{N}_0)$ such that*

$$\|(i\lambda_n - A)\| \leqq t_n \qquad \text{for all} \quad n \geqq k_0,\, k_0 \in \mathbb{N}_0.$$

Then, for each $f \in \mathrm{AP}_\Lambda(\mathbb{R}, H)$ there is a unique mild solution of (1.1) *in* $\mathrm{AP}_\Lambda(\mathbb{R}, H)$.

Proof. By a result from [3] which is given in the scalar case, but whose proof is also valid in a Banach space, Conditions (i) and (ii) on $\sigma(f)$ imply that

$$f(t) = \sum_{k=0}^{\infty} a(\lambda_k, f) e^{i\lambda_k t}, \tag{1.4}$$

where the convergence is uniform on $\mathbb{R}$.

From the fact that H is a Hilbert space, it is known (see [8]) that Parseval's equation holds; that is,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|f(s)\|^2\, ds = \sum_{k=0}^{\infty} \|a(\lambda_k, f)\|^2. \tag{1.5}$$

We define

$$U_N(t) = \sum_{k=0}^{N} (i\lambda_k - A)^{-1}\, a(\lambda_k, f)\, e^{i\lambda_k t}, \qquad N \in \mathbb{N},\, t \in \mathbb{R} \tag{1.6}$$

$$f_N(t) = \sum_{k=0}^{N} a(\lambda_k, f) e^{i\lambda_k t}. \tag{1.7}$$

Then the U_N are strict a.p. solutions of

$$U_N'(t) = A U_N(t) + f_N(t), \tag{1.8}$$

i.e.,

$$U_N(t) = T_{t-s} U_N(s) + \int_s^t T_{t-v} f_N(v)\, dv, \qquad s \leqq t,\, N \in \mathbb{N}. \tag{1.9}$$

Moreover, for M, $N \in \mathbb{N}$, we have

$$\begin{aligned} \|U_N(t) - U_M(t)\| &= \left\| \sum_{k=M+1}^{N} (i\lambda_k - A)^{-1} a(\lambda_k, f) e^{i\lambda_k t} \right\| \\ &\leqq \sum_{k=M+1}^{N} |t_k| \, \|a(\lambda_k, f)\| \\ &\leqq \left(\sum_{k=M+1}^{N} |t_k|^2 \right)^{1/2} \left(\sum_{k=M+1}^{N} \|a(\lambda_k, f)\|^2 \right)^{1/2}. \end{aligned}$$

Hence, U_N converges uniformly on $\mathbb{R}$ to a function u; in particular (see [8]), $u \in \mathrm{AP}_A(\mathbb{R}, H)$.

Observation. If we compare the Sup-norm of f with respect to the Sup-norm of the solution of (1.1) found through the previous process, we have

$$\|u\|_\infty \leqq \|(t_n)\|_2 \, \|f\|_\infty. \tag{1.10}$$

THEOREM 1.3. *Let A be the infinitesimal generator of a C_0-semigroup on the Hilbert space H such that $A = B + C$, where B and C are linear operators with $D(B) \subseteq D(C)$ and satisfying*

$$\|Cx\| \leqq a\,\|x\| + b\,\|Bx\|; \qquad 0 \leqq b < 1/2,\ a > 0,\ x \in D(B) \tag{1.11}$$

$$\sum_{a,b} = \{i\lambda/\lambda \in \mathbb{R}, |\lambda| > a/(1-2b)\} \subseteq \rho(B) \tag{1.12}$$

$$\|(zI - B)^{-1}\| \leqq \frac{M}{|z|}; \qquad z \in \sum_{a,b}; M > 0. \tag{1.13}$$

Let $f \in \mathrm{AP}_A(\mathbb{R}, H)$. The following conditions are equivalent:

(a) *There exists a mild* a.p. *solution of* (1.1).

(b) *For $|\lambda_k| \leqq a/(1-2b)$ we have $a(\lambda_k, f) \in \mathrm{Im}(i\lambda_k - A)$.*

Proof. By Theorem 1.1 we known that (a) implies (b); conversely, we define

$$\begin{aligned} f_0(t) &= \sum_{|\lambda| \leqq a/(1-2b)} a(\lambda_k, f) e^{i\lambda_k t} \\ f_1(t) &= \sum_{|\lambda_k| > a/(1-2b)} a(\lambda_k, f) e^{i\lambda_k t}. \end{aligned} \tag{1.14}$$

Then, from (1.5),

$$f(t) = f_0(t) + f_1(t), \qquad t \in \mathbb{R}. \tag{1.15}$$

By hypothesis, for each λ_k with $|\lambda_k| \leqq a/1-2b$, there exists $x_k \in D(A)$ such that

$$a(\lambda_k, f) = (i\lambda_k - A)x_k \tag{1.16}$$

$$u_0(t) = \sum_{|\lambda_k| \leqq a/(1-2b)} x_k e^{i\lambda_k t}. \tag{1.17}$$

From (1.13) and (1.16) we obtain that $u_0(t)$ is a strict a.p. solution of the equation

$$u'(t) = Au(t) + f_0(t), \qquad t \in \mathbb{R}. \tag{1.18}$$

On the other hand, if $|\lambda_k| > a/(1-2b)$, then from (1.12) and (1.13) we have

$$i\lambda_k \in \rho(B) \tag{1.19}$$

and

$$\|(i\lambda_k - B)^{-1}\| \leqq \frac{M}{|\lambda_k|}. \tag{1.20}$$

Moreover, from (1.11), we get for $x \in H$

$$\|C(i\lambda_k - B)^{-1} x\| \leqq a\, \|(i\lambda_k - B)^{-1} x\| + b\, \|B(i\lambda_k - B)^{-1} x\|. \tag{1.21}$$

Making use of (1.20) and the identity $B(i\lambda_k - B)^{-1} = i\lambda_k(i\lambda_k - B)^{-1} - I$, we obtain

$$\|C(i\lambda_k - B)^{-1} x\| \leqq \frac{aM}{|\lambda_k|} \|x\| + bM\, \|x\| + b\, \|x\|, \qquad x \in H. \tag{1.22}$$

We consider the following two cases:

Case 1. $0 \leqq M < 1$. Then, from (1.22), we obtain $\|C(i\lambda_k - B)^{-1}\| \leqq a/|\lambda_k| + 2b$.

Case 2. $M > 1$. Then, dividing by M in (1.22), we obtain $(1/M)\, \|C(i\lambda_k - B)^{-1}\| \leqq a/|\lambda_k| + 2b$.

Since the only point of accumulation of $\{\lambda_k\}_k$ is ∞, there exists a $\beta > 0$ such that

$$\frac{1}{|\lambda_k|} < \frac{1-2b}{a} - \beta; \tag{1.23}$$

therefore, if we define, for $\delta \in \{1, M\}$, $\|x\|_\delta = (1/\delta)\, \|x\|$, $x \in H$, $\delta > 0$, we have

$$\|C(i\lambda_k - B)^{-1}\|_\delta \leqq \frac{a}{|\lambda_k|} + 2b < 1 - \beta a < 1. \tag{1.24}$$

Then $(I-C(i\lambda_k-B)^{-1})^{-1}$ exists and

$$(I-C(i\lambda_k-B)^{-1})^{-1}=\sum_{k=0}^{\infty}(C(i\lambda_k-B)^{-1})^k, \tag{1.25}$$

where convergence is taken in terms at the norm $\|\cdot\|_\delta$. On the other hand,

$$\begin{aligned}(I-C(i\lambda_k-B)^{-1})^{-1}(i\lambda_k-B)^{-1}&=[(i\lambda_k-B)(I-C(i\lambda_k-B)^{-1})]^{-1}\\&=(i\lambda_k-B-C)^{-1}=(i\lambda_k-A)^{-1};\end{aligned}$$

then, from (1.20) and (1.25), we have

$$\begin{aligned}\|(i\lambda_k-A)^{-1}\|_\delta&\leqq\frac{\|(i\lambda_k-B)^{-1}\|_\delta}{1-\|C(i\lambda_k-B)^{-1}\|_\delta}\\&\leqq\frac{M\delta^{-1}}{|\lambda_k|}\frac{1}{1-\|C(i\lambda_k-B)^{-1}\|_\delta}, \qquad \delta\in\{1,M\}.\end{aligned} \tag{1.26}$$

Observe that (1.24) yields

$$\beta a<1-\|C(i\lambda_k-B)^{-1}\|_\delta, \qquad \delta\in\{1,M\}, \tag{1.27}$$

so that, from (1.26), we have

$$\|(i\lambda_k-A)^{-1}\|_\delta<\frac{M\delta^{-1}}{|\lambda_k|}\frac{1}{\beta a}, \qquad \delta\in\{1,M\}. \tag{1.28}$$

Now, we observe that Condition (ii) on $\sigma(f)$ implies

$$|\lambda_0|\leqq\Theta\,|\lambda_1|\leqq\Theta^2\,|\lambda_2|\leqq\cdots\leqq\Theta^k\,|\lambda_k|; \qquad \Theta\in(0,1) \tag{1.29}$$

Let $k_0=\min\{k/\lambda_k\neq 0\}$; then, from (1.29),

$$\frac{1}{|\lambda_k|}\leqq\frac{\Theta^{k-k_0}}{|\lambda_{k_0}|}, \qquad \Theta\in(0,1), k\geqq k_0. \tag{1.30}$$

Hence, from (1.28), we obtain

$$\|(i\lambda_k-A)^{-1}\|_\delta\leqq\frac{\Theta^{-k_0}}{|\lambda_{k_0}|}\cdot\frac{M\cdot\delta}{\beta\cdot a}\Theta^k, \qquad \Theta\in(0,1); \delta\in\{1,M\}; k\geqq k_0.$$

Finally, in either case,

$$\|(i\lambda_k-A)^{-1}\|<\frac{\Theta^{-k_0}}{|\lambda_{k_0}|}\cdot\frac{M}{\beta\cdot a}\cdot\Theta^k \qquad \Theta\in(0,1), k\geqq k_0,$$

and the result is a consequence of Theorem 1.2.

Remark. The previous theorem holds in the following cases:

(1) A a linear bounded operator (compare with [7] in the periodic case);

(2) B self-adjoint, C satisfying (1.11), $D(B) \subseteq D(C)$, and $A = B + C$ the generator of a C_0-semigroup; in particular, if C is bounded (compare [7]).

(3) B self-adjoint and the infinitesimal generator of a C_0-semigroup, moreover C a bounded operator.

(4) B the infinitesimal generator of a bounded analytic C_0-semigroup of angle α (see [9]), C a closed linear operator which satisfies (1.11), $D(B) \subseteq D(C)$.

Let $\{\lambda_n\}_{n \geqq 1}$, $\lambda_n \neq 0$, be a sequence of real numbers which satisfies conditions (i) and (ii). Denote $\lambda_0 = 0$ and define $\Lambda_0 = \{\lambda_n\}_{n \geqq 0}$.

The following result generalizes [7].

COROLLARY. *Let A be the infinitesimal generator of a C_0-semigroup on a Hilbert space H such that*

$$i\Lambda_0 \backslash \{0\} \subseteq \rho(A) \tag{1.31}$$

$$\|(i\lambda_n - A)^{-1}\| \leqq \frac{M}{|\lambda_n|}, \qquad n \geqq 1. \tag{1.32}$$

Let $f \in \mathrm{AP}_{\Lambda_0}(\mathbb{R}, H)$. Then the following conditions are equivalent:

(a) *There exists a mild* a.p. *solution of* (1.1).

(b) $\lim_{T \to \infty} (1/T) \int_0^T f(t)\, dt \in \operatorname{Im}(A)$.

Proof. By hypothesis and as in the proof of Theorem 1.2, we have

$$f(t) = \sum_{k=0}^{\infty} a(\lambda_k, f) e^{i\lambda_k t},$$

where the convergence is uniform on $\mathbb{R}$. Moreover, there exists $x \in D(A)$ such that $a(0, f) = -Ax$.

Define $f_0(t) = -Ax$, $f_1(t) = \sum_{k=1}^{\infty} a(\lambda_k, f) e^{i\lambda_k t}$. Following the proof of Theorem 1.3 with $a = 0$ and $C \equiv 0$, we have that $u_0(t) = x$ is the solution of (1.18), and using (1.20) and (1.30) we have

$$\|(i\lambda_k - A)^{-1}\| \leqq \frac{M}{|\lambda_1|} \cdot \Theta^{k-1}, \qquad \Theta \in (0, 1), k = 1, 2, \ldots;$$

then by Theorem 1.2 with $\Lambda = \Lambda_0 \setminus \{0\}$ and (1.31), there exists for $f_1(t)$ a unique mild solution a.p. of $u'(t) = Au(t) + f_1(t)$. So, the corollary is valid.

Application to the Schrödinger Equation. Consider on $H = L^2(\mathbb{R}^n)$ the Schrödinger equation,

$$\frac{\partial u}{\partial t} = i \cdot \Delta u - i \cdot Vu.$$

We apply Theorem 1.3 with $B = i \cdot \Delta$ and $C = iV$. From [11, Chap. 7, Lemma 5.4 and Theorem 5.5] it is known that, if $V \in L^p(\mathbb{R}^n)$ with $p \geqq 2$, $p > n/2$, for each $\varepsilon > 0$ we have

$$\|Vu\| \leqq \varepsilon \|\Delta u\| + k\varepsilon^\alpha \|u\|, \qquad u \in H^2(\mathbb{R}^n) = D(iV),$$

where k is a constant independent of ε, $\alpha > 0$.

Moreover, $A = B + C = i\Delta - iV$ is the infinitesimal generator of a C_0-semigroup on $L^2(\mathbb{R}^n)$.

On the other hand, it is known (see [9, A-IV, Example 1.15]) that Δ generates a bounded analytic C_0-semigroup on $L^2(\mathbb{R}^n)$, therefore (see [9, A-II, Theorem 1.14]) there exist $M > 0$ and $r \geqq 0$ such that

$$iz \in \rho(B) \quad \text{and}$$

$$\|R(iz, B)\| \leqq \frac{M}{|z|} \quad \text{if} \quad \operatorname{Re} z > 0 \quad \text{and} \quad |z| \geqq r.$$

Now, choose $0 \leqq \varepsilon < \frac{1}{2}$ such that $k\varepsilon^\alpha/(1 - 2\varepsilon) > r$; then if $\lambda \in \mathbb{R}$ and $|\lambda| > k\varepsilon^\alpha/(1 - 2\varepsilon)$, we have that

$$i\lambda \in \rho(B) \quad \text{and} \quad \|(i\lambda - B)^{-1}\| \leqq \frac{M}{|\lambda|},$$

so that Conditions (1.12) and (1.13) of Theorem 1.3 are satisfied. Hence, for any $f \in \mathrm{AP}_\Lambda(\mathbb{R}, H)$, the equation

$$\frac{\partial u}{\partial t} = i \cdot \Delta u - iVu + f$$

has a unique mild a.p. solution in $\mathrm{AP}_\Lambda(\mathbb{R}, H)$ if and only if for each $\lambda_k \in \sigma(f)$ with $|\lambda_k| < k\varepsilon^\alpha/(1 - 2\varepsilon)$ we have $a(\lambda_k, f) \in \operatorname{Im}(i\lambda_k - i\Delta + iV)$.

2. The Case of the Second Order Problem

Let A be the infinitesimal generator of a strongly continuous cosine function. In this section, the second Cauchy abstract problem is considered:

$$u''(t) = Au(t) + f(t), \tag{2.1}$$

where f is an a.p. function.

Note that $u \in C^1(\mathbb{R}, H)$ is called a mild solution of (2.1) if it satisfies the formula

$$u(t) = C_{t-s}u(s) + S_{t-s}u'(s) + \int_s^t S_{t-v}f(v)\,dv, \qquad s, t \in \mathbb{R}, \tag{2.2}$$

where $t \to S_t$ is the associated sine function defined by

$$S_t x = \int_0^t C_s x\,ds. \tag{2.3}$$

If $u \in C^2(\mathbb{R}, H)$ satisfies (2.1), then it is called a strict solution.

The next result is the analogue of Theorem 1.1 of the first section and generalizes Lemma 1 in [2].

THEOREM 2.1. *Let A be the infinitesimal generator of a strongly continuous cosine function defined in a Banach space X. If $f \in \mathrm{AP}(\mathbb{R}, X)$ and $u(t)$ is a mild* a.p. *solution (of class C^1) of* (2.1). *Then*

$$a(\lambda, u) \in D(A) \qquad \text{and} \qquad (\lambda^2 + A)\,a(\lambda, u) = -a(\lambda, f), \qquad \lambda \in \mathbb{R}.$$

Proof. Since u is a mild solution of (2.1), from (2.2) we have

$$C_s u(t) + S_s u'(t) = u(t+s) - \int_t^{t+s} S_{t+s-v} f(v)\,dv, \qquad s, t \in \mathbb{R}. \tag{2.4}$$

Moreover, it is easy to see that the following identities are valid:

$$C_s a(\lambda, u) = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-i\lambda v} C_s u(v)\,dv, \qquad \lambda \in \mathbb{R}; \tag{2.5}$$

$$a(\lambda, u') = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-i\lambda v} u'(v)\,dv = i\lambda a(\lambda, u'), \qquad \lambda \in \mathbb{R}. \tag{2.6}$$

Using (2.5) and (2.6) we have

$$C_s a(\lambda, u) + S_s a(\lambda, u') = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda v}\{C_s u(v) + S_s u'(v)\}\, dv$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda v}\left\{u(v+s) - \int_v^{v+s} S_{v+s-t} f(t)\, dt\right\} dv$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda v} u(v+s)\, dv$$

$$- \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda v}\int_v^{v+s} S_{v+s-t} f(t)\, dt\, dv$$

$$= e^{i\lambda s} a(\lambda, u) - \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda v}\int_0^s S_{s-u} f(v+u)\, du\, dv$$

$$= e^{i\lambda s} a(\lambda, u) - e^{i\lambda s}\int_0^s e^{-i\lambda v} S_v a(\lambda, f)\, dv.$$

Using (2.6) again we have

$$C_s a(\lambda, u) + i\lambda S_s a(\lambda, u) = e^{i\lambda s} a(\lambda, u) - e^{i\lambda s}\int_0^s e^{-i\lambda v} S_v a(\lambda, f)\, dv; \qquad (2.7)$$

then

$$\frac{2}{s^2}(C_s - I)\, a(\lambda, u) + \frac{2}{s^2}(i\lambda S_s - i\lambda s)\, a(\lambda, u)$$

$$= \frac{2}{s^2}(e^{i\lambda s} - I - i\lambda s)\, a(\lambda, u) - \frac{2}{s^2} e^{i\lambda s}\int_0^s e^{-i\lambda v} S_v a(\lambda, f)\, dv.$$

We let $s \to 0$ and observe that in this case

$$\frac{2}{s^2}(i\lambda S_s x - i\lambda s x) \to 0$$

$$\frac{2}{s^2}(e^{i\lambda s} x - x - i\lambda s x) \to -\lambda^2 x$$

$$\frac{2e^{i\lambda s}}{s^2}\int_0^s e^{-i\lambda v} S_v x\, dv \to x.$$

THEOREM 2.2. *Let H be a Hilbert space and A the infinitesimal generator of a strongly continuous cosine function. Suppose that $\{-\lambda_k^2\} \subseteq \rho(A)$ and that there exists $\{t_k\}_{k \geqq 0}$ in $l^2(\mathbb{N}_0)$ such that $\|\lambda_k(\lambda_k^2 + A)^{-1}\| \leqq t_k$ for all $k \geqq k_0$. Then, for each $f \in \mathrm{AP}_\Lambda(\mathbb{R}, H)$, there exist a unique mild solution of (2.1) (of class C^1) $u(t) \in \mathrm{AP}_\Lambda(\mathbb{R}, H)$. Moreover, $u'(t)$ is also almost periodic.*

THEOREM 2.3. *Let A be the infinitesimal generator of a strongly continuous cosine function on a Hilbert space H. Suppose that $A = B + C$, where B and C are linear operators with $D(B) \subseteq D(C)$ and such that the following conditions are satisfied:*

$$\|Cx\| \leqq a\,\|x\| + b\,\|Bx\| \qquad \text{for some } 0 \leqq b < 1/2,\ a > 0,\ x \in D(B) \tag{2.8}$$

$$\sum_{a,b} = \{-\lambda^2/\lambda \in \mathbb{R}, \qquad |\lambda| > a/(1-2b)\} \subseteq \rho(B) \tag{2.9}$$

$$\|\lambda(\lambda^2 + B)^{-1}\| \leqq M/|\lambda|, \qquad \lambda \in \mathbb{R},\ |\lambda| > a/(1-2b),\ M > 0. \tag{2.10}$$

Let $f \in \mathrm{AP}_\Lambda(\mathbb{R}, H)$; the following conditions are equivalent:

(a) *There exists a unique mild* a.p. *solution with $u'(t)$* a.p. *of* (2.1).

(b) *For $\lambda_k^2 \leqq a/(1-2b)$, we have $a(\lambda_k, f) \in \mathrm{Im}(\lambda_k^2 + A)$.*

The proofs of the above two theorems are similar to those of Theorems 2.2 and 2.3 and therefore we omit them.

Remark. Theorem 2.3 holds, for example, when A is a bounded linear operator, or when B generates a cosine function which satisfies the spectral conditions (2.9) and (2.10) and C is a bounded operator (see [10]).

Application to the Wave Equation. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u. \tag{2.11}$$

Let us consider in Theorem 2.3 $A = \Delta$ defined on $H = L^2(\mathbb{R}^n)$.

It is known that Δ is the infinitesimal generator of a strongly continuous cosine function (see [5]). Therefore we have

$$-\lambda^2 \in \rho(A) \qquad \text{and} \qquad \|\lambda(\lambda^2 + A)^{-1}\| \leqq \frac{M}{|\lambda|} \qquad \text{if} \quad |\lambda| \geqq \sqrt{r}$$

so that conditions (2.9) and (2.10) are satisfied with $a = \sqrt{r}$ and $b = 0$. We conclude that for $f \in \mathrm{AP}_\Lambda(\mathbb{R}, H)$ there exists a unique mild a.p. solution in $\mathrm{AP}_\Lambda(\mathbb{R}, H)$ of

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + f$$

if and only if for $\lambda_k \in \sigma(f)$ with $\lambda_k^2 \leqq \sqrt{r}$ we have that

$$a(\lambda_k, f) \in \mathrm{Im}(\lambda_k^2 + \Delta).$$

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